Math 115A, Lecture 2
Linear Algebra

Midterm 2

Instructions: You have 50 minutes to complete the exam. There are five problems, worth a total of fifty points. You may not use any books or notes. Partial credit will be given for progress toward correct proofs.

Write your solutions in the space below the questions. If you need more space use the back of the page. Do not forget to write your name in the space below.

Name: ______________________________

Question	Points	Score
1	10	
2	10	
3	10	
4	10	
5	10	
Total:	50	

Problem 1.

Let $T : V \to W$ be a linear transformation between real vector spaces V and W.

(a) [5pts.] Define $R(T)$ and $N(T)$.

(b) [5pts.] Prove that T is one-to-one if and only if $N(T) = \{0\}$.

Problem 2.

Consider the linear transformation

$$\begin{aligned} T : P(\mathbb{R}) &\to P(\mathbb{R}) \\ f(x) &\mapsto f'(x) + f(0) \end{aligned}$$

(a) [3pts.] Is T onto?

(b) [3pts.] Is T an isomorphism?

(c) [4pts.] What are the eigenvectors of T?

Problem 3.

Let T be the linear transformation of the plane given by rotating $\frac{\pi}{4}$ radians counterclockwise and then reflecting across the x-axis.

(a) [5pts.] Find the matrix representing $[T]_\beta$ representing T with respect to the standard basis β for $\mathbb{R}^2$.

(b) [5pts.] Find a basis β' for $\mathbb{R}^2$ with respect to which T is represented by a diagonal matrix.

Problem 4.

Let V be a finite-dimensional vector spaces, and let $\mathcal{L}(V)$ be the vector space of linear transformations from V to itself.

(a) [5pts.] Consider the subset Z of $\mathcal{L}(V)$ consisting of the invertible linear transformations from V to itself. Is Z a subspace of $\mathcal{L}(V)$?

(b) [5pts.] Show that $\beta = \{T_1, \cdots, T_n\}$ is a basis for $\mathcal{L}(V)$ and $S \in \mathcal{L}(V)$ is invertible, then $S(\beta) = \{ST_1, \cdots, ST_n\}$ is also a basis for $\mathcal{L}(V, W)$. [Hint: You will want to use the fact that S^{-1} exists.]

Problem 5.

Let $T : V \to V$ be a linear transformation.

(a) [5pts.] Prove that if λ is an eigenvalue of T, then λ^n is an eigenvalue of T^n.

(b) [5pts.] Suppose that $\lambda > 0$ is an eigenvalue of T^n. Is $\lambda^{\frac{1}{n}}$ necessarily an eigenvalue of T? [Hint: Think about rotations.]